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ENERGY RELEASE CAUSED BY THE KINKING OF A CRACK IN A PLANE ANISOTROPIC SOLID†

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The problem of the deformation of a homogeneous, elastic, anisotropic body with a rectilinear edge cut having a small kink Υ_τ of fairly arbitrary shape is analysed. The asymptotic solution of this problem in the case of small values of the dimensionless parameter τ, characterizing the kink size is constructed using a modified method of matched asymptotic expansions. The amount of elastic energy released is expressed in terms of a set of stress intensity factors (SIFs) at the tip of the unperturbed crack, the integral characteristics of the kink Υ_τ (the components of the enlarged energy release matrix) and the integral characteristics of the initial solid (the SIFs of the weighting functions). The results are compared with existing results in the case of an isotropic solid. © 2002 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider a homogeneous, anisotropic, elastic solid Ω_0 which has a rectilinear edge cut Ξ_0. We will assume that the solid is loaded on the external boundary Γ under conditions of plane deformation. The displacement vector $\mathbf{u}^0 = (u_1^0, u_2^0)$ satisfies the problem

$$L(\nabla_x)\mathbf{u}^0(\mathbf{x}) = 0, \quad \mathbf{x} = (x_1, x_2) \in \Omega_0 \tag{1.1}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{u}^0; \mathbf{x}) = \mathbf{p}^0(\mathbf{x}), \ \ \mathbf{x} \in \Gamma; \ \ \boldsymbol{\sigma}^{(n)}(\mathbf{u}^0; \mathbf{x}) = 0, \ \ \mathbf{x} \in \Xi_0^+ \cup \Xi_0^- \tag{1.2}$$

Here $L(\nabla_x)$ is the (anisotropic) Lamé operator, $\boldsymbol{\sigma}^{(n)}$ is the stress vector in the plane with unit normal $\mathbf{n}$, which is outward with respect to Ω_0, and Ξ_0^+ and Ξ_0^- are the upper and lower edges of the cut Ξ_0. The load $\mathbf{p}^0$ is assumed to be self-balanced, that is

$$\int_\Gamma p_k^0(\mathbf{x})ds_x = 0, \quad k = 1, 2, \quad \int_\Gamma [x_1 p_2^0(\mathbf{x}) - x_2 p_1^0(\mathbf{x})]ds_x = 0 \tag{1.3}$$

Suppose l is the shortest distance from the tip O of the crack Ξ_0 to the boundary Γ. We denote a small positive parameter by τ and determine the crack increment Υ_τ. We introduce the "stretched" coordinates

$$\boldsymbol{\xi} = \tau^{-1}\mathbf{x}$$

In the plane of the variables (ξ_1, ξ_2), we produce a piecewise-smooth simple arc Υ, which is covered by a circle of diameter l. The arc Υ_τ is obtained by compressing Υ τ^{-1} times, that is

$$\Upsilon_\tau = \{(x_1, x_2): \ \tau^{-1}(x_1, x_2) \in \Upsilon\}$$

Finally, on removing the small set Υ_τ from Ω_0, we obtain the domain Ω_τ with an increased and, generally speaking, kinked out Ξ_τ (Fig. 1).

The vector of the displacements of the points of the solid $\mathbf{u}^\tau$ satisfies the relations

$$L(\nabla_x)\mathbf{u}^\tau(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\tau \tag{1.4}$$

$$\begin{aligned} &\boldsymbol{\sigma}^{(n)}(\mathbf{u}^\tau; \mathbf{x}) = \mathbf{p}^0(\mathbf{x}), \quad \mathbf{x} \in \Gamma \\ &\boldsymbol{\sigma}^{(n)}(\mathbf{u}^\tau; \mathbf{x}) = 0, \quad \mathbf{x} \in \Xi_\tau^+ \cup \Xi_\tau^- \end{aligned} \tag{1.5}$$

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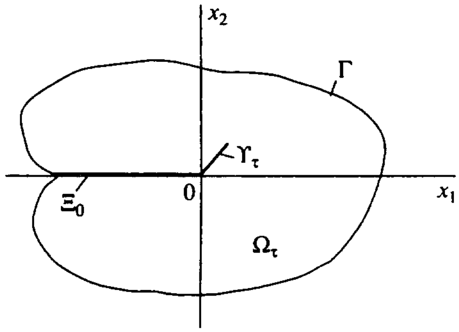

Fig. 1

By Clapeyron's theorem, the elastic potential energy of deformation of a solid Ω_τ is given by the following formula (the dot denotes a scalar product)

$$U(\Omega_\tau;\mathbf{u}^\tau)=-\frac{1}{2}\int_\Gamma \mathbf{p}^0(\mathbf{x})\cdot\mathbf{u}^\tau(\mathbf{x})ds_x \tag{1.6}$$

We will investigate the behaviour of the solution of problem (1.4), (1.5) and the functional (1.6) for small values of the parameter τ.

A numerical solution of the problem of the fracture of an initially rectilinear crack in an anisotropic, elastic plane has already been obtained ([1, 2], etc.). Assuming a slight inclination of the kink, an approximate solution was found in [5] by the small-parameter method [3, 4]. In the isotropic case, asymptotic formulae for the stress intensity factors (SIFs) were also derived in [6, 7] and elsewhere. The asymptotic form of the energy release has been constructed [8, 9, etc.]. The problem of the kinking of a crack in an elastic solid with finite dimensions was considered in [10, 11]. The most complete results were obtained in [11, 12].

In this paper, the shape of the kink Υ_τ is not constrained by any assumptions. The main result of the paper is the complete asymptotic expansion of the energy (1.6) with respect to the parameter τ.

2. THE BASIS OF THE POWER SOLUTIONS IN THE PROBLEM OF A SEMI-INFINITE CRACK

An asymptotic representation of the vector $\mathbf{u}^0(x)$ when $r = |\mathbf{x}| \to 0$ is formed from the power solutions $r^\Lambda\mathbf{\Phi}(\varphi, \ln r)$ in which Λ is a complex number, $\varphi \in (-\pi, \pi)$ is the polar angle and $\mathbf{\Phi}$ is a polynomial of the variable $\ln r$ with smooth coefficients with respect to φ. It has been established in [13] (and, besides, these facts have long been known in special cases of anisotropy) that the exponent is an integer or half-integer and that $\ln r$ only appears when $\Lambda = 0$.

In accordance with the mechanical interpretation, it is customary to subdivide the above-mentioned solutions into four groups. The solutions

$$\mathbf{X}^{j,2m+1}(r,\varphi)=r^{m+1/2}\mathbf{\Phi}^{j,2m+1}(\varphi),\quad j=1,2;\quad m=0,1,\ldots \tag{2.1}$$

belong to the first group. These solutions possess a finite elastic energy in any circle and generate singularities in the stresses or their derivatives at the crack tip. The vectors (2.1) satisfy the following normalization conditions on the kink of the crack

$$\sigma_{i2}(\mathbf{X}^{j,2m+1};x_1,0)=(2\pi)^{-1/2}r^{m-1/2}\delta_{3-i,j},\quad x_1>0,\ i=1,2 \tag{2.2}$$

The second group contains homogeneous polynomials of the variables x_1 and x_2

$$\mathbf{X}^{j,2m}(r,\varphi)=r^m\mathbf{\Phi}^{j,2m}(\varphi),\quad j=1,2;\quad m=0,1,\ldots \tag{2.3}$$

The vectors $\mathbf{X}^{1,0} = \mathbf{e}^1$, $\mathbf{X}^{2,0} = \mathbf{e}^2$ and $\mathbf{X}^{2,2}$ $(r, \varphi) = x_1\mathbf{e}^2 - x_2\mathbf{e}^1$ correspond to translational motions and rotation. The other polynomial solutions (2.3) can be subject to the same normalization conditions

$$\sigma_{11}(\mathbf{X}^{j,2m}; x_1, 0) = r^{m-1}\delta_{1,j}, \quad x_1 > 0, \quad m = 1, 2, \ldots$$
$$\partial_2\sigma_{11}(\mathbf{X}^{j,2m}; x_1, 0) = -(m-1)r^{m-2}\delta_{2,j}, \quad x_1 > 0, \quad m = 2, 3, \ldots \tag{2.4}$$

Henceforth, the simplified notation $\partial_j = \partial/\partial x$ is used for derivatives and $\delta_{i,j}$ is the Kronecker delta. The remaining solutions have singularities and, in the case of these solutions, the energy integral diverges at the cut tip. For instance, the vectors

$$\mathbf{Y}^{j,2m+1}(r,\varphi) = r^{-m-1/2}\boldsymbol{\Psi}^{j,2m+1}(\varphi), \quad j = 1, 2; \quad m = 0, 1, \ldots \tag{2.5}$$

$$\mathbf{Y}^{j,0}(r,\varphi) = \boldsymbol{\psi}^{j,0}(\varphi)\ln r + \boldsymbol{\Psi}^{j,0}(\varphi), \quad j = 1, 2$$
$$\mathbf{Y}^{j,2m}(r,\varphi) = r^{-m}\boldsymbol{\Psi}^{j,2m}(\varphi), \quad j = 1, 2; \quad m = 1, 2, \ldots \tag{2.6}$$

occur in the third and fourth groups.

Note that the solutions $\mathbf{Y}^{1,0}$, $\mathbf{Y}^{2,0}$ and $\mathbf{Y}^{2,2}$, which are paired with $\mathbf{X}^{1,0}$, $\mathbf{X}^{2,0}$ and $\mathbf{X}^{2,2}$, generate forces and moments that are concentrated at the crack tip. We normalize the non-energy solution (2.5) and (2.6) with the conditions [14]

$$q(\mathbf{X}^{j,k}, \mathbf{Y}^{i,n}; \gamma) = \delta_{i,j}\delta_{n,k} \tag{2.7}$$
$$q(\mathbf{u}, \mathbf{v}; \gamma) = \int_\gamma [\boldsymbol{\sigma}^{(n)}(\mathbf{u}; \mathbf{x})\cdot\mathbf{v}(\mathbf{x}) - \boldsymbol{\sigma}^{(n)}(\mathbf{v}; \mathbf{x})\cdot\mathbf{u}(\mathbf{x})]ds_x$$

Here γ is an arbitrary arc, encompassing the tip, that is, which has ends on the opposite edges of the semi-infinite crack. By virtue of the Betti identity, the left-hand side of equality (2.7) is independent of γ.

Remark 1. In the case of an isotropic material, the angular parts $\boldsymbol{\Phi}^{\cdots}$ and $\boldsymbol{\Psi}^{\cdots}$ have been indicated [14] when normalizations (2.2) and (2.7) are satisfied, but with a change in normalization (2.4). In the case of general anisotropy, it has been verified [15] that the power solutions may be subject to requirements (2.2) and (2.7).

We will now show that the polynomial solutions (2.3) allow of normalization (2.4). If this were not so, a polynomial solution $\mathbf{X}(\mathbf{x}) = r^m\boldsymbol{\Phi}(\varphi)$ would be found for a certain m, for which

$$\sigma_{11}(\mathbf{X}; x_1, 0) = 0, \quad \partial_2\sigma_{11}(\mathbf{X}; x_1, 0) = 0, \quad x_1 > 0 \tag{2.8}$$

In the case of the polynomials, equalities (2.8) extend to all values of x_1. Similarly, the boundary conditions on the sides of a semi-infinite cut

$$\sigma_{12}(\mathbf{X}; x_1, 0) = 0, \quad \sigma_{22}(\mathbf{X}; x_1, 0) = 0, \quad x_1 < 0 \tag{2.9}$$

also extend along the whole abscissa axis.

Relations (2.8) and (2.9) can be differentiated with respect to x_1. Hence, from the equilibrium equations, which are satisfied for polynomials over the whole plane

$$\partial_2\sigma_{21}(\mathbf{X}; \mathbf{x}) = -\partial_1\sigma_{11}(\mathbf{X}; \mathbf{x}) = 0, \quad \partial_2\sigma_{22}(\mathbf{X}; \mathbf{x}) = -\partial_1\sigma_{12}(\mathbf{X}; \mathbf{x}) \tag{2.10}$$

we initially derive the formulae $\partial_2\sigma_{2i}(\mathbf{X}; x_1, 0) = 0$ ($i = 1, 2$) and, then, differentiating Eqs (2.10) with respect to x_2, also the formulae $\partial_2^2\sigma_{2i}(\mathbf{X}; x_1, 0) = 0, \partial_2^3\sigma_{22}(\mathbf{X}; x_1, 0) = 0$. So, for a certain $n \geqslant 2$,

$$\sigma_{11}(\mathbf{X}; \mathbf{x}) = x_2^n P_{11}(\mathbf{x}), \quad \sigma_{12}(\mathbf{X}; \mathbf{x}) = x_2^{n+1}P_{12}(\mathbf{x}), \quad \sigma_{22}(\mathbf{X}; \mathbf{x}) = x_2^{n+2}P_{22}(\mathbf{x}) \tag{2.11}$$

The degree of the polynomial P_{ij} is equal to $m - n - i - j + 2$. We assume that the number n is taken to be as large as possible in equalities (2.11), and this means that the coefficient c in the polynomial P_{11} is non-zero when x_1^{m-n}. Actually, if $c = 0$, then

$$\partial_2^k\sigma_{11}(\mathbf{X}; x_1, 0) = 0, \quad \partial_2^k\sigma_{12}(\mathbf{X}; x_1, 0) = 0, \quad \partial_2^{k+1}\sigma_{22}(\mathbf{X}; x_1, 0) = 0, \quad k = 0, \ldots, n$$

and, using Eqs (2.10), we observe that $\partial_2^{n+1}\sigma_{12}(\mathbf{X}; x_1, 0) = \partial_2^{n+2}\sigma_{22}(\mathbf{X}; x_1)$, that is, we increase n by unity. We now put $C = c(m-n)!$ and obtain

$$\sigma_{11}(\partial_1^{m-n}\mathbf{X};\mathbf{x})=Cx_2^n,\quad \sigma_{12}(\partial_1^{m-n}\mathbf{X};\mathbf{x})=\sigma_{22}(\partial_1^{m-n}\mathbf{X};\mathbf{x})=0$$

The deformations $\varepsilon_{ik}(\partial_1^{m-n}\mathbf{X};\,\mathbf{x})$ satisfy the compatibility equation

$$0=\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}-2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}=Ca_{11}n(n-1)x_2^{n-2}$$

only in the case when $C = c = 0$ since $n \geqslant 2$ and a_{11} is a non-zero diagonal element of the pliability matrix. The necessary contradiction has been found.

The asymptotic form of the solution of problem (1.1), (1.2) in the neighbourhood of the crack tip can be written as

$$\mathbf{u}^0(\mathbf{x})=\sum_{(j,n)}c_{j,n}^0\mathbf{X}^{j,n}(r,\varphi)+O(r^{(N+1)/2}),\quad r\to 0 \tag{2.12}$$

Any natural N can be taken in relation (2.12). We shall subsequently fix this, and summation must be carried out over $j = 1, 2, n = 1, \ldots, N$. On account of the fact that the coefficients $c_{1,0}^0$, $c_{2,0}^0$ and $c_{2,2}^0$ are arbitrary in the case of stiff displacements, they can be taken equal to zero and the summation contracted by eliminating the corresponding pairs of indices. The coefficients $c_{1,1}^0$ and $c_{2,1}^0$ are the SIFs K_1^0 and K_2^0.

Remark 2. Normalization (2.2) conforms with the definition of the SIF which is used in strong criteria, and it therefore follows that the basis of the power solutions which has been introduced should be referred to as a "strong" basis. The concept of the opening of a crack arises in the deformation criteria and the "deformation" basis is therefore associated with the normalization conditions

$$[X_\varphi^{j,2m+1}]=-r^{m+1/2}\delta_{1,j},\quad [X_r^{j,2m+1}]=-r^{m+1/2}\delta_{2,j} \tag{2.13}$$

$$X_r^{j,2m}(r,0)=r^m\delta_{1,j},\quad X_\varphi^{j,2m}(r,0)=r^m\delta_{j,2} \tag{2.14}$$

($[X] = X(r, +\pi) - X(r, -\pi)$ is the jump in the function X on the crack sides).

Note that relation (2.14) was used previously in [14] instead of (2.4).

The possibility of complying with normalizations (2.13) and (2.14) is proved indirectly. For example, denying the possibility that requirements (2.13) can be satisfied, we find a power solution $\mathbf{X}(\mathbf{x}) = r^{m+1/2}\mathbf{\Phi}^m(\varphi)$ for which $[X_i] = 0$ $(i = 1, 2)$. Since, $[\sigma_{2i}(\mathbf{X})] = 0$ in view of the boundary conditions on the cut sides, the vector $\mathbf{X}$ satisfies the homogeneous equilibrium equations everywhere in the plane with the exception of the cut tip. Consequently, the exponent Λ accompanying it must be an integer and not a half-integer $m + 1/2$.

3. DIFFERENTIATION ALONG A CRACK

Since $L(\nabla_x)$ is a differential operator with constant coefficients (the elastic material is homogeneous), every power solution $U(x) = r^\Lambda\mathbf{\Phi}(\varphi)$ which differs from a constant remains a power solution after carrying out the differentiation $\partial/\partial x_1$, but has an exponent $\nabla - 1$. It is easily shown that the vectors (2.1) and (2.3) are related by the equalities

$$\partial_1\mathbf{X}^{j,n+2}(r,\varphi)=\frac{n}{2}\mathbf{X}^{j,n}(r,\varphi),\quad n=0,1,2,\ldots \tag{3.1}$$

Note that the factor $m - 1$ in the second relation of (2.4) is, in fact, set up precisely in order to satisfy (3.1).

The formula

$$q(\partial_1\mathbf{U},\mathbf{V};\gamma)=-q(\mathbf{U},\partial_1\mathbf{V};\gamma) \tag{3.2}$$

for the power vectors $\mathbf{U}$ and $\mathbf{V}$ has been verified in [16].

When account is taken of (2.7), the relation

$$\partial_1 \mathbf{Y}^{j,n}(r,\varphi) = -\frac{n}{2}\mathbf{Y}^{j,n+2}(r,\varphi), \quad n = 0, 1, 2, \ldots \tag{3.3}$$

follows from equalities (3.1) and (3.2).

Hence, in order to construct the basis vectors (2.1) and (2.5), it is sufficient to calculate the angular parts $\boldsymbol{\Phi}^{j,1}$ and $\boldsymbol{\Psi}^{j,1}$ and to comply with the normalization conditions (2.2) and (2.7). The remaining vectors of (2.1) and (2.5) for $m = 1, 2, \ldots$ are determined in accordance with equalities (3.1) and (3.3). Moreover, the expansion

$$-\partial_1 \mathbf{X}^{j,1}(r,\varphi) = \alpha_{j1}\mathbf{Y}^{1,1}(r,\varphi) + \alpha_{j2}\mathbf{Y}^{2,1}(r,\varphi), \quad j = 1, 2 \tag{3.4}$$

holds. The coefficients α_{jk} are determined in terms of the constants of elasticity using the formula

$$\alpha_{jk} = -q(\mathbf{X}^{k,1}, \ \partial_1 \mathbf{X}^{j,1}), \quad j, k = 1, 2 \tag{3.5}$$

Here, the matrix $\boldsymbol{\alpha} = \|\alpha_{jk}\|$ is found to be symmetric (see (3.2)) and positive-definite (see [15]). The quantities α_{jk} depend on the crack orientation with respect to the axes of anisotropy. We shall assume that $\alpha_0 = (\alpha_{11} + \alpha_{22})/2$.

The relation between the form of q from conditions (2.7) and the invariant Cherepanov–Rice integral

$$J(\mathbf{u}^0;\gamma) = -\frac{1}{2}q(\mathbf{u}^0, \partial_1\mathbf{u}^0;\gamma) \tag{3.6}$$

$$J(\mathbf{u};\gamma) = \int_\gamma [W(\mathbf{u};\mathbf{x})\cos(\mathbf{n}, x_1) - \boldsymbol{\sigma}^{(n)}(\mathbf{u};\mathbf{x})\cdot\partial_1\mathbf{u}(\mathbf{x})]ds_x$$

has been established in [16].

Here $W(\mathbf{u};\,\mathbf{x})$ is the elastic energy density corresponding to the displacement field $\mathbf{u}$. Hence, by relations (2.7), (3.4)–(3.6) and (2.12), we have

$$J(\mathbf{u}^0;\gamma) = \frac{1}{2}[\alpha_{11}(K_1^0)^2 + 2\alpha_{12}K_1^0K_2^0 + \alpha_{22}(K_2^0)^2] \tag{3.7}$$

4. WEIGHTING FUNCTIONS

We will call the pairs of indices (j, n), which differ from (1,0), (2,0) and (2,2), permissible pairs. Following the well-known approach in [17, 18], we introduce weighting functions for the permissible pairs of indices, that is, the non-energy solutions of the homogeneous problem (1.1), (1.2) with singularities at the cut tip,

$$\boldsymbol{\zeta}^{j,n}(\mathbf{x}) = \mathbf{Y}^{j,n}(r,\varphi) + O(1), \quad r \to 0 \tag{4.1}$$

The normalization conditions (2.7) ensure [18, 16] integral representations for the coefficients in expansion (2.12)

$$c_{j,n}^0 = \int_{\Gamma_\sigma} \mathbf{p}^0(\mathbf{x})\cdot\boldsymbol{\zeta}^{j,n}(\mathbf{x})ds_x \tag{4.2}$$

The asymptotic formula (4.1) can be refined to give

$$\boldsymbol{\zeta}^{j,n}(\mathbf{x}) = \mathbf{Y}^{j,n}(r,\varphi) + \sum_{(i,p)} m_{i,p}^{j,n}\mathbf{X}^{i,p}(r,\varphi) + O(r^{(N+1)/2}), \quad r \to 0 \tag{4.3}$$

Here, as in Section 2, we assume that $m_{1,0}^{j,n} = m_{2,0}^{j,n} = m_{2,2}^{j,n} = 0$, that is, the summation is carried out over the permitted pairs (i, p). From the coefficients $m_{i,p}^{j,n}$, we set up the $(2N-1)\times(2N-1)$ SIF matrix of the weighting functions $m = \|m_{i,p}^{j,n}\|$, coordinating the numbering of the rows and columns in the following manner

$$\begin{Vmatrix} m_{1,1}^{1,1} & m_{2,1}^{1,1} & m_{1,2}^{1,1} & m_{1,3}^{1,1} & m_{2,3}^{1,1} & \dots & m_{1,N}^{1,1} & m_{2,N}^{1,1} \\ m_{1,1}^{2,1} & m_{2,1}^{2,1} & m_{1,2}^{2,1} & m_{1,3}^{2,1} & m_{2,3}^{2,1} & \dots & m_{1,N}^{2,1} & m_{2,N}^{2,1} \\ m_{1,1}^{1,2} & m_{2,1}^{1,2} & m_{1,2}^{1,2} & m_{1,3}^{1,2} & m_{2,3}^{1,2} & \dots & m_{1,N}^{1,2} & m_{2,N}^{1,2} \\ m_{1,1}^{1,3} & m_{2,1}^{1,3} & m_{1,2}^{1,3} & m_{1,3}^{1,3} & m_{2,3}^{1,3} & \dots & m_{1,N}^{1,3} & m_{2,N}^{1,3} \\ m_{1,1}^{2,3} & m_{2,1}^{2,3} & m_{1,2}^{2,3} & m_{1,3}^{2,3} & m_{2,3}^{2,3} & \dots & m_{1,N}^{2,3} & m_{2,N}^{2,3} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{1,1}^{1,N} & m_{2,1}^{1,N} & m_{1,2}^{1,N} & m_{1,3}^{1,N} & m_{2,3}^{1,N} & \dots & m_{1,N}^{1,N} & m_{2,N}^{1,N} \\ m_{1,1}^{2,N} & m_{2,1}^{2,N} & m_{1,2}^{2,N} & m_{1,3}^{2,N} & m_{2,3}^{2,N} & \dots & m_{1,N}^{2,N} & m_{2,N}^{2,N} \end{Vmatrix} \qquad (4.4)$$

The symmetry and the positive definiteness of the matrix m (see [14, §1.5]) is established by applying Green's formula to the functions $\zeta^{j,n}$ and $\zeta^{i,p}$ in the domain Ω_0 from which a circle of radius ε has been removed, together with taking the limit as $\varepsilon \to 0$. The coefficients $m_{i,p}^{j,n}$ are defined solely by the geometry of the Ω_0 domain and depend on the constants of elasticity, rather than on the applied load $\mathbf{p}^0$. If L has the dimension of length, the dimension of the quantity $\alpha_0 m_{i,p}^{j,n}$ is equal to $L^{-(p+n)/2}$.

5. THE ENERGY RELEASE MATRIX

We will now consider the plane of the stretched coordinates and, in the domain $G = \mathbf{R}^2 \setminus (\Lambda_0 \cup \Upsilon)$, where Λ_0 is the ray $\{\boldsymbol{\xi}: \xi_1 \leqslant 0, \xi_2 = 0\}$, we consider the elastic problem

$$L(\nabla_\xi)\mathbf{w}(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in G; \quad \boldsymbol{\sigma}^{(n)}(\mathbf{w}; \boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \partial G \qquad (5.1)$$

Here ∂G is the boundary of the domain G, consisting of the cut sides $\Lambda_0^+ \cup \Upsilon^+$ and $\Lambda_0^- \cup \Upsilon^-$.

As earlier in [14], we denote the special solutions of the homogeneous problem (5.1), which increase at infinity as $\mathbf{X}^{j,n}(\rho, \varphi) = \rho^{n/2} \boldsymbol{\Phi}^{j,n}(\varphi)$, where $\rho = \varepsilon^{-1} r$ is the "stretched" polar radius, by $\boldsymbol{\eta}^{j,n}$. The expansion

$$\boldsymbol{\eta}^{j,n}(\boldsymbol{\xi}) = \mathbf{X}^{j,n}(\rho, \varphi) + \sum_{(i,p)} M_{i,p}^{j,n} \mathbf{Y}^{i,p}(\rho, \varphi) + O(\rho^{-(N+1)/2}) \qquad (5.2)$$

holds as $\rho \to \infty$.

We now introduce the $(2N-1) \times (2N-1)$ matrix $M = \|M_{i,p}^{j,n}\|$, ordering the coefficients of expansion (5.2) in it in the same way as in (4.4). It has been established [14], that the matrix M is symmetric and non-negative-definite (positive-definite, if, for example, Υ is not an interval that extends the ray Λ_0). The coefficients $M_{i,p}^{j,n}$ depend on the constants of elasticity and are determined by the shape and size of the kink Υ. The dimensionality of the quantity $\alpha_0^{-1} M_{i,p}^{j,n}$ is equal to $L^{(n+p)/2}$.

Remark 3. If Υ is an interval, that extends the cut Λ_0, then, in the case of even n, the solution (5.2) is identical to the polynomial $\mathbf{X}^{j,n}$, and this means that the rows and columns in the matrix M, corresponding to the pairs (j, n), turn out to contain zeros (in this case, M is only a non-negative-definite matrix). We will denote the remaining 2×2 cells with elements $M_{q,2p+1}^{i,2m+1}$ $(i, q = 1, 2)$ by M_{2p+1}^{2m+1} and derive recurrence formulae for them using the method proposed earlier in [19]. Actually, we represent the solution $\boldsymbol{\eta}^{j,2m+1}$ as a linear combination of power solutions

$$\mathbf{h}^{j,2k+1}(\rho, \varphi) = \mathbf{X}^{j,2k+1}(\rho_l, \varphi_l), \quad k = 0, \dots, m \qquad (5.3)$$

referred to the polar coordinates ρ_l and $\varphi_l \in (-\pi, \pi)$ with centre at the tip of the cut $\Lambda_0 \cup \Upsilon$. We put

$$S(\mathbf{h}^{\dots,2m+1}) = \sqrt{2\pi} \begin{Vmatrix} \sigma_{22}(\mathbf{h}^{1,2m+1}; \rho, 0) & \sigma_{22}(\mathbf{h}^{2,2m+1}; \rho, 0) \\ \sigma_{12}(\mathbf{h}^{1,2m+1}; \rho, 0) & \sigma_{12}(\mathbf{h}^{2,2m+1}; \rho, 0) \end{Vmatrix}$$

By relations (2.2) and (3.4), (3.3), we have

$$S(\mathbf{X}^{\dots,2k+1}) = \rho^{k-1/2} E_2, \quad S(\mathbf{Y}^{\dots,2p+1}) = \frac{1}{2} \rho^{-p-3/2} \alpha^{-1} \qquad (5.4)$$

where E_2 is an identity 2×2 matrix and α is a matrix with elements (3.5). Using Taylor's formula, we obtain the expansion

$$S(\mathbf{h}^{\cdots,2m+1}) = E_2\rho_l^{m-1/2} = E_2(\rho-l)^{m-1/2} = E_2\rho^{m-1/2}\left\{1+\sum_{j=1}^{m}\frac{(2m-1)!!}{[2(m-j)-1]!!}\frac{1}{j!}\left(-\frac{l}{2\rho}\right)^j + \right.$$

$$\left. +(2m-1)!!\left(-\frac{l}{2\rho}\right)^m\sum_{k=1}^{N}\frac{(2k-1)!!}{(k+m)!}\left(\frac{l}{2\rho}\right)^k + O(\rho^{-N-3/2})\right\}$$

($(2k-1)!!$ is the product of the odd numbers $1, 3, \ldots, 2k-1$, where $(-1)!! = 1$). These relations enable one to determine the coefficients of the above-mentioned linear combinations and, moreover, together with equalities (5.4), lead to the recurrence formula

$$M_{2p+1}^{2m+1} + \sum_{j=1}^{m}\frac{(2m-1)!!}{[2(m-j)-1]!!}\frac{1}{j!}\left(-\frac{l}{2}\right)^j M_{2p+1}^{2(m-j)+1} = 2\alpha\frac{(-1)^m}{(m+p+1)!}(2m-1)!!(2p+1)!!\left(\frac{l}{2}\right)^{m+p+1}$$

In particular, taking $m = p = 0$, we find the well-known relation [15]

$$M_{1,1}^{1,1} = \alpha_{11}l, \quad M_{2,1}^{1,1} = M_{1,1}^{2,1} = \alpha_{12}l, \quad M_{2,1}^{2,1} = \alpha_{22}l \tag{5.5}$$

We emphasize that, in the case of a linear increase in the cut, the non-zero cells of the matrix M differ from the 2×2 matrix α only in numerical factors, which are independent of the form of the anisotropy.

6. THE MODIFIED METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

As applied to problems of the brittle fracture mechanics, the method of matched asymptotic expansions [20, 21, 22, etc.] has been developed in detail [23, 14]. The idea of a modification of this method which increases the accuracy of the asymptotic solution [25] was apparently put forward for the first time in [24].

For fixed N, we take the sum

$$\mathbf{v}(r;\ \mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \sum_{(j,n)} a_{j,n}\boldsymbol{\zeta}^{j,n}(\mathbf{x}) \tag{6.1}$$

as the outer asymptotic expansion of the field $\mathbf{u}^\tau(\mathbf{x})$ at a distance from the mouth of the crack Ξ_τ, where $a_{j,n}$ are coefficients to be determined.

Close to the mouth of the kink Υ_τ, we approximate $\mathbf{u}^\tau(\mathbf{x})$ by a linear combination with unknown coefficients

$$\mathbf{w}(\tau;\mathbf{x}) = \sum_{(j,n)} b_{j,n}\boldsymbol{\eta}^{j,n}(\tau;\mathbf{x}) \tag{6.2}$$

Since the inner asymptotic expansion (6.2) is written in real coordinates, instead of asymptotic formula (5.2), the following is required

$$\boldsymbol{\eta}^{j,n}(\tau;\mathbf{x}) = \mathbf{X}^{j,n}(r,\varphi) + \sum_{(i,p)} M_{i,p}^{j,n}(\tau)\mathbf{Y}^{i,p}(r,\varphi) + O(r^{-(N+1)/2}) \tag{6.3}$$

The components $M_{i,p}^{j,n}(\tau)$ of the matrix $M(\tau)$ for the kink Υ_τ appear here, where

$$M_{i,p}^{j,n}(\tau) = \tau^{(n+p)/2}M_{i,p}^{j,n} \tag{6.4}$$

According to relations (2.12) and (4.3), the expansion

$$\mathbf{v}(\tau;x) = \sum_{(j,n)} c_{j,n}^0\mathbf{X}^{j,n}(r,\varphi) + \sum_{(j,n)} a_{j,n}\mathbf{Y}^{j,n}(r,\varphi) +$$

$$+\sum_{(j,n)}\sum_{(i,p)} a_{j,n}m_{i,p}^{j,n}\mathbf{X}^{i,p}(r,\varphi) + O(r^{(N+1)/2}), \quad r\to 0 \tag{6.5}$$

holds.

On the other hand, by virtue of formulae (6.3) and (6.4), the relations

$$\mathbf{w}(\tau;\mathbf{x}) = \sum_{(j,n)} b_{j,n}\mathbf{X}^{j,n}(r,\varphi) + \sum_{(j,n)}\sum_{(i,p)} b_{j,n}M_{i,p}^{j,n}(\tau)\mathbf{Y}^{i,p}(r,\varphi) + O(\tau^{(N+1)/2}r^{-(N+1)/2}) \qquad (6.6)$$

for the inner asymptotic expansion (6.2) when $\tau^{-1}r \to \infty$ are satisfied.

Suppose $\mathbf{c}^0$, $\mathbf{a}$ and $\mathbf{b}$ are columns composed of the coefficients $c_{j,n}^0$, $a_{j,n}$ and $b_{j,n}$ with the permissible pairs of indices. The matching of the outer and inner expansions (6.5) and (6.6) implies that the individual asymptotic terms are identical. The relations which arise in this case form a system of linear equations in the columns $\mathbf{a}$ and $\mathbf{b}$

$$c^0 + m\mathbf{a} = \mathbf{b}, \quad \mathbf{a} = M(\tau)\mathbf{b} \qquad (6.7)$$

Denoting a $2N-1$ identity matrix by E, we find the solution of system (6.7)

$$\mathbf{a} = [E - M(\tau)m]^{-1}M(\tau)\mathbf{c}^0, \quad \mathbf{b} = [E - mM(\tau)]^{-1}\mathbf{c}^0 \qquad (6.8)$$

7. THE ASYMPTOTIC FORMULA FOR THE INCREMENT IN THE DEFORMATION POTENTIAL ENERGY

Substituting its outer asymptotic expansion (6.1) instead of the vector $\mathbf{u}^\tau$, we have

$$U(\Omega_\tau;\mathbf{u}^\tau) = U(\Omega_0;\mathbf{u}^0) - \frac{1}{2}\sum_{(j,n)} a_{j,n}\int_{\Gamma_\sigma}\mathbf{p}^0(\mathbf{x})\cdot\boldsymbol{\zeta}^{j,n}(\mathbf{x})ds_x + O(\tau^{N+1/2})$$

Evaluating the integrals using formula (4.2), we obtain

$$\Delta U = U(\Omega_\tau;\mathbf{u}^\tau) - U(\Omega_0;\mathbf{u}^0) = -\frac{1}{2}\sum_{(j,n)} a_{j,n}c_{j,n}^0 + O(\tau^{N+1/2})$$

The rigorous proof of the formula which has been obtained follows from the general results in [26] (also, see [22, 27]).

Taking the first relation of (6.8) into account, we obtain

$$\Delta U = -\frac{1}{2}(\mathbf{c}^0)^T[E - M(\tau)m]^{-1}M(\tau)\mathbf{c}^0 + O(\tau^{N+1/2}) \qquad (7.1)$$

The superscript T means transposition. We emphasize that the matrix m is composed of the SIFs of the weighting functions and is defined by the geometry of the initial solid Ω_0. Following the well-known approach [19], integral representations can be derived for the coefficients $m_{i,k}^{j,n}$.

When $N = 1$, formula (7.1) is simplified and becomes (compare with [15])

$$\Delta U = -\frac{1}{2}(K_1^0, K_2^0)\begin{Vmatrix} M_{1,1}^{1,1}(\tau) & M_{2,1}^{1,1}(\tau) \\ M_{1,1}^{2,1}(\tau) & M_{2,1}^{2,1}(\tau)\end{Vmatrix}\begin{pmatrix} K_1^0 \\ K_2^0\end{pmatrix} + O(\tau^{3/2}) \qquad (7.2)$$

The main 2×2 block $\mathbf{M}(\tau) = \|M_{i,1}^{j,1}(\tau)\|$ of the matrix $M(\tau)$ which appears in (7.2) is appropriately called the *elastic energy release matrix* when a small additional cut Υ appears in the initial solid Ω_0.

If Υ_τ is a segment of length $l_\tau = \tau l$, which extends the cut Ξ_0, then the matrix $\mathbf{M}(\tau)$, according to relations (5.5), is replaced by $l_\tau\alpha$ and formula (7.2) is an extension of the well-known Irwin formula [28]

$$\Delta U = -\frac{1}{2}l_\tau(\mathbf{K}^0)^T\boldsymbol{\alpha}\mathbf{K}^0 + O(\tau^{3/2}), \quad \mathbf{K}^0 = (K_1^0, K_2^0)^T \qquad (7.3)$$

In this case, the energy release rate is equal to

$$G = \lim_{\tau\to 0}\frac{U(\Omega_0;\mathbf{u}^0) - U(\Omega_\tau;\mathbf{u}^\tau)}{l_\tau} = \frac{1}{2}[\alpha_{11}(K_1^0)^2 + 2\alpha_{12}K_1^0K_2^0 + \alpha_{22}(K_2^0)^2] \qquad (7.4)$$

We recall (see relations (7.4) and (3.7)), that the quantity G is identical to the invariant Cherepanov–Rice integral $J(\mathbf{u}^0;\gamma)$, evaluated using the solution of the initial problem (1.1), (1.2).

We determine the values of the constants α_{ij} by comparing formula (7.4) with the results in [29]. We write the relation between the components of the strain and stress tensors in the form [30]

$$\varepsilon_{ii}=a_{i1}\sigma_{11}+a_{i2}\sigma_{22}+a_{i6}\sigma_{12},\quad i=1,2;\quad 2\varepsilon_{12}=a_{16}\sigma_{11}+a_{26}\sigma_{22}+a_{66}\sigma_{12}$$

Then, according to the calculations in [29], we obtain

$$\alpha_{11}=-\frac{a_{11}}{2}\operatorname{Im}(\mu_1+\mu_2)\bar{\mu}_1\bar{\mu}_2,\quad \alpha_{12}=-\frac{a_{11}}{2}\operatorname{Im}\bar{\mu}_1\bar{\mu}_2,\quad \alpha_{22}=\frac{a_{11}}{2}\operatorname{Im}(\mu_1+\mu_2)$$

Here $\operatorname{Im}\mu_i>0$, and μ_1 and μ_2 are the roots of the characteristic equation [30]

$$a_{11}\mu^4-2a_{16}\mu^3+(2a_{12}+a_{66})\mu^2-2a_{26}\mu+a_{22}=0$$

8. THE ASYMPTOTIC STRESS INTENSITY FACTORS AT THE TIP OF A CRACK WITH A LINEAR KINK

Suppose Υ_τ is an interval of length $l_\tau=\tau l$ starting from the tip of the crack Ξ at an angle $\beta\in(-\pi,\pi)$. At the kink vertex we introduce the polar coordinates $\hat{r}$ and $\hat{\varphi}\in(-\pi,\pi)$, directing the polar axis along Υ_τ. The expansion

$$\mathbf{u}^\tau(\mathbf{x})=c_{1,0}^\tau\hat{\mathbf{X}}^{1,0}+c_{2,0}^\tau\hat{\mathbf{X}}^{2,0}+K_1^\tau\hat{\mathbf{X}}^{1,1}(\hat{r},\hat{\varphi})+K_2^\tau\hat{x}^{2,1}(\hat{r},\hat{\varphi})+O(\hat{r}) \tag{8.1}$$

holds for the displacement vector $\mathbf{u}^\tau(\mathbf{x})$ when $\hat{r}\to 0$. We emphasize that, in the case of an arbitrary anisotropy when $\beta\neq 0$, the expressions for the vectors $\hat{\mathbf{X}}^{j,1}(\hat{r},\hat{\varphi})$ differ from $\mathbf{X}^{j,1}(\hat{r},\hat{\varphi})$.

The asymptotic forms of the SIFs K_1^τ and K_2^τ at the tip of the crack which has grown are determined using the inner asymptotic expansion (6.2). We present the formulae for the special solutions, similar to (8.1),

$$\boldsymbol{\eta}^{j,n}(\tau;\mathbf{x})=K_1^{j,n}(\tau)\hat{\mathbf{X}}^{1,1}(\hat{r},\hat{\varphi})+K_2^{j,n}(\tau)\hat{\mathbf{X}}^{2,1}(\hat{r},\hat{\varphi})+O(\hat{r}) \tag{8.2}$$

$$K_i^{j,n}(\tau)=\tau^{(n-1)/2}K_i^{j,n},\quad i=1,2 \tag{8.3}$$

Hence, by relations (6.2) and (8.2), (8.3), we find the asymptotic expansions for the SIFs K_1^τ and K_2^τ from expansion (8.1)

$$K_i^\tau\simeq\sum_{(j,n)}b_{j,n}K_i^{j,n}(\tau)=\sum_{(j,n)}\tau^{(n-1)/2}b_{j,n}K_i^{j,n} \tag{8.4}$$

Here $b_{j,n}$ are the elements of a column given by the second formula of (6.8), and $K_1^{j,n}$ and $K_2^{j,n}$ are coefficients (corresponding to the value $\tau=1$) which are independent of the parameter τ.

When $N=3$, formula (8.4), with an accuracy $O(\tau^{3/2})$, appears as

$$K_i^\tau\simeq\sum_{j=1}^{2}K_j^0K_i^{j,1}+\tau^{1/2}T_0K_i^{1,2}+$$
$$+\tau\left\{\sum_{j=1}^{2}k_j^0K_i^{j,3}+\sum_{j=1}^{2}K_i^{j,1}[K_1^0(m_{1,1}^{j,1}M_{1,1}^{1,1}+m_{2,1}^{j,1}M_{1,1}^{2,1})+K_2^0(m_{1,1}^{j,1}M_{2,1}^{1,1}+m_{2,1}^{j,1}M_{2,1}^{2,1})]\right\} \tag{8.5}$$

Here, $K_j^0=c_{j,1}^0$, $T_0=c_{1,2}^0$ and $k_j^0=c_{j,3}^0$ are the SIF, the tension intensity along the crack and the lower SIFs, and $m_{1,1}^{j,n}$ and $m_{2,1}^{j,n}$ are the SIFs of the weighting function $\zeta^{j,n}$. The basis of the asymptotic representations of the SIFs is obtained, as usual, using methods developed previously [22, 27, 31].

Methods proposed earlier [15] enable one to relate the quantities $M_{i,k}^{j,n}$ and the SIFs of the special solution $\boldsymbol{\eta}^{j,n}(\xi)$ and $\boldsymbol{\eta}^{i,k}(\xi)$

$$M_{i,k}^{j,n}=\frac{2l}{k+n}(K_1^{i,k},K_2^{i,k})\begin{Vmatrix}\hat{\alpha}_{11} & \hat{\alpha}_{12}\\ \hat{\alpha}_{21} & \hat{\alpha}_{22}\end{Vmatrix}\begin{pmatrix}K_1^{j,n}\\ K_2^{j,n}\end{pmatrix} \tag{8.6}$$

Here $\hat{\alpha}_{j1}$, $\hat{\alpha}_{j2}$ are coefficients in the formula, similar to (3.4), for the differentiation of the vector $\hat{\mathbf{X}}^{j,1}(\hat{r}, \hat{\varphi})$ along the kind Υ.

Now, using relation (8.6) the asymptotic formula (7.2) for the energy increment can be rewritten in the form [15]

$$\Delta U = -\frac{1}{2} l_\tau (\mathbf{K}^\tau)^T \hat{\alpha} \mathbf{K}^\tau + O(\tau^{3/2}) \tag{8.7}$$

Here $l_\tau = \tau l$ is the length of the kink Υ_τ, $\mathbf{K}^\tau = (K_1^\tau, K_2^\tau)^T$ is the SIF column vector at the tip of Υ_τ and $\hat{\alpha}$ is a matrix which appears in formula (8.6).

In the case of an isotropic body $\alpha_{12} = 0$ and $\alpha_{11} = \alpha_{22} = \alpha_0 = (4\mu)^{-1}(1 + \varkappa)$, where $\varkappa = (\lambda + 3\mu)(\lambda + \mu)^{-1}$, and λ and μ are Lamé constants. Here, relation (7.3) is identical to the classical Irwin formula [28]

$$\Delta U = -\frac{1}{2} l_\tau \alpha_0 \,|\, \mathbf{K}^0 \,|^2 + O(\tau^{3/2}) \tag{8.8}$$

In this case, relation (8.6) is simplified

$$M_{i,k}^{j,n} = \frac{2\alpha_0 l}{k+n} (K_1^{i,k} K_1^{j,n} + K_2^{i,k} K_2^{j,n}) \tag{8.9}$$

Setting up the 2×2 matrix $F = \|F_{ij}\|$ from the coefficients $K_i^{j,1} = F_{ij}$, we rewrite formula (8.9) when $n = k = 1$ in the matrix form

$$\mathbf{M} = \alpha_0 l F^T F \tag{8.10}$$

Substituting the quantities $M_{i,1}^{j,1}$, calculated from (8.10), into (8.5), we obtain the result in [11] for the case of a straight kink Υ_τ. Expansions in powers of the parameter $m = \beta/\pi$ have been obtained for the quantities $K_i^{j,1}$ and $l^{-1/2} K_i^{1,2}$ in [12]. Using the results in [12] and relation (8.9), the coefficients $M_{i,1}^{j,1}$ are easily calculated.

9. DISCUSSION OF THE RESULTS AND REMARKS

1. In the case of an anisotropic solid, the surface energy density which appears in the fracture energy criterion depends on the direction of the development of the crack (see [32], etc.). Suppose a crack Ξ_0 is situated in the most dangerous direction, that is, $\min \gamma = \gamma(0)$. We will now find the necessary conditions for the rectilinearity of its development. The increment in the total energy, caused by the nucleation of the kink $\Upsilon_\tau(\beta)$ in the form of a segment of length $l_\tau = \tau l$ directed at an angle β to Ξ_0, is, by virtue of (7.2), equal to

$$2\gamma(\beta)\tau l - \frac{1}{2}(\mathbf{K}^0)^T \mathbf{M}(\tau;\beta)\mathbf{K}^0 = \tau\left[2\gamma(\beta) l - \frac{1}{2}(\mathbf{K}^0)^T \mathbf{M}(1;\beta)\mathbf{K}^0\right] \tag{9.1}$$

Here, $\mathbf{M}(\tau;\beta)$ is the energy release matrix for the kink $\Upsilon_\tau(\beta)$.

According to Griffith's criterion [33], a crack grows rectilinearly subject to the condition that the angle $\beta = 0$ corresponds to the *global* minimum of the quantity (9.1). Since, according to the assumption, $\partial_\beta \gamma(0) = 0$, the quantity (9.1) has a *local* minimum in the direction $\beta = 0$ when the two relations

$$(\mathbf{K}^0)^T \partial_\beta \mathbf{M}(1;0)\mathbf{K}^0 = 0 \tag{9.2}$$

$$4\partial_\beta^2 \gamma(0) l - \frac{1}{2}(\mathbf{K}^0)^T \partial_\beta^2 \mathbf{M}(1;0)\mathbf{K}^0 > 0 \tag{9.3}$$

are satisfied. The first of them requires that the matrix $\partial_\beta \mathbf{M}(1;0)$ should not be of fixed sign and leads to a relation between the SIF column with the characteristic numbers μ_i and the columns m_i of this matrix

$$K^0 = \text{const}(|\,\mu_2\,|^{1/2} m_1 \pm |\,\mu_1\,|^{1/2} m_2) \tag{9.4}$$

Since $\partial_\beta^2\gamma(0) \geqslant 0$, inequality (9.3) is guaranteed, for example, in the case of a negative-definite matrix $\partial_\beta^2\mathbf{M}(1;0)$. We emphasize that the above-mentioned necessary conditions have to do with the first and second derivatives of the matrix $\mathbf{M}(1;0)$, rather than with the matrix itself.

Using the calculations in [12], for an isotropic solid we have

$$\partial_\beta \mathbf{M}(1;0) = \frac{\alpha_0 l}{2}\begin{Vmatrix} 0 & -1 \\ -1 & 0 \end{Vmatrix}, \quad \partial_\beta^2 \mathbf{M}(1;0) = \frac{\alpha_0 l}{2}\operatorname{diag}\left\{-1, \left(3-\frac{16}{\pi^2}\right)\right\}$$

$$m_1 = (1,\ 1)^T, \quad \mu_1 = 1, \quad m_2(-1,\ 1)^T, \quad \mu_2 = -1$$

Consequently, by virtue of expression (9.4), equality (9.2) is possible in two situations: a pure first mode $K_1^0 = C, K_2^0 = 0$ and a pure second mode $K_1^0 = 0, K_2^0 = C$. The second diagonal element of the matrix $\partial_\beta^2\mathbf{M}(1;0)$ is positive, and this means the impossibility of a rectilinear development of a crack under shear loading.

2. According to what has been said in Section 2, it makes sense to furnish the deformation and force criteria with different bases $\mathbf{X}_{(d)}^{j,n}$ and $\mathbf{X}_{(s)}^{j,n}$ of the power solutions. Suppose the transition from the pair $\mathbf{X}_{(d)}^{1,1}$, $\mathbf{X}_{(d)}^{2,1}$ to the pair $\mathbf{X}_{(s)}^{1,1}$, $\mathbf{X}_{(s)}^{2,1}$ is made using the 2×2 matrix $T = \|T_{ij}\|$. Comparing normalizations (2.2) and (2.13), we see that the same matrix is suitable for the pairs $\mathbf{X}_{(d)}^{1,2m+1}$, $\mathbf{X}_{(d)}^{2,2m+1}$ and $\mathbf{X}_{(s)}^{1,2m+1}$, $\mathbf{X}_{(s)}^{2,2m+1}$ with an arbitrary natural number m. By virtue of conditions (2.7) for the singular solutions $\mathbf{Y}_{\dots}^{1,2m+1}$, $\mathbf{Y}_{\dots}^{2,2m+1}$, it is necessary to use the inverse matrix T^{-1}. Finally, on account of conditions (2.13), the condition for the opening of the cut Ξ_0 in terms of the SIFs (a force basis) is written as

$$K_1^0 T_{11} + K_2^0 T_{12} \geqslant 0$$

3. The basis $\mathbf{X}_{(e)}^{j,n}$ of power solutions adapted to the energy criteria for fracture was introduced in [15]. The matrix which accomplishes the transition from the pair $\mathbf{X}_{(s)}^{1,1}$, $\mathbf{X}_{(s)}^{2,1}$ to the pair $\mathbf{X}_{(e)}^{1,1}$, $\mathbf{X}_{(e)}^{2,1}$ is equal to $\alpha_0^{1/2}\boldsymbol{\alpha}^{-1/2}$, where α_0 and $\boldsymbol{\alpha}$ are defined in Section 3 (see formulae (3.4) and (3.5)). We set up the column $\mathbf{K}_{(e)}^0$ from the coefficients $c_{1,1}^0$ and $c_{2,1}^0$ in an asymptotic expansion, similar to (2.12), with respect to the energy basis of the solution of the unperturbed problem $\mathbf{u}^0$. The column $\mathbf{K}_{(e)}^0$ is related to the SIF column $\mathbf{K}^0$ as follows [15]:

$$\mathbf{K}_{(e)}^0 = \alpha_0^{-1/2}\boldsymbol{\alpha}^{1/2}\mathbf{K}^0$$

Hence, on expressing the vector $\mathbf{K}^0$ in terms of $\mathbf{K}_{(e)}^0$ and substituting into expression (7.3), we find

$$\Delta U = -\frac{1}{2} l_\tau \alpha_0 \,|\mathbf{K}_{(e)}^0|^2 + O(\tau^{3/2}) \tag{9.5}$$

As a result, formula (9.5) for the energy increment for an anisotropic material in the case of the rectilinear propagation of a crack does not differ in form from the classical Irwin formula (8.9).

In the case of a segment Υ_τ of length l_τ making an angle β with the direction of the crack Ξ_0, formula (7.2) becomes

$$\Delta U = -\frac{1}{2} l_\tau \alpha_0 (\mathbf{K}_{(e)}^0)^T \tilde{\mathbf{M}}(\beta)\mathbf{K}_{(e)}^0 + O(\tau^{3/2})$$

Here, $\tilde{\mathbf{M}}(\beta) = (l_\tau\alpha_0)^{-1}\boldsymbol{\alpha}^{-1/2}\mathbf{M}(\tau;\beta)\boldsymbol{\alpha}^{-1/2}$ is a symmetric matrix with dimensionless coefficients, which is normalized by the condition $\tilde{\mathbf{M}}(0) = E_2$. The properties which have been enumerated make the normalized elastic energy release matrix $\tilde{\mathbf{M}}(\beta)$ a canonical characteristic of the kinking of a semi-infinite crack.

4. The enlarged elastic energy release matrix $M(\tau)$ is defined by the shape and size of the kink Υ_τ. Integral representations can be obtained [14] for the coefficients $M_{i,k}^{j,n}$. We emphasize that the values of the coefficients $M_{i,k}^{j,n}$ depend on the basis of the power solutions used. Actually, the transition from a force basis to a deformation or energy basis must be accompanied by a recalculation of the components of

the matrix M. Earlier [34, 35, 36, etc.], integral characteristics, similar to the matrix M, were introduced for other problems in mechanics and other geometrical situations.

The proposed asymptotic procedure is also applicable in th case of a family of edge and internal cracks (compare with the investigation in [37] of the tensile crack). Note that the interaction of cracks is described using an analogue of matrix (4.4), which is made up of the SIFs of the weighting functions at all the tips and therefore has dimensions which increase in proportion to the number of tips. Such a matrix preserves its symmetry but, generally speaking, loses its positive definiteness (see [19]).

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